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X. Investigation of an Extensive Class of Partial Differential Equations of the Second Order, in which the Equation of LAPLACE'S Functions is included.

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Theorem. IF u be a function of x and y satisfying the equation

$$\frac{d^2u}{dx dy} + \alpha_n e^\phi u = 0,$$

where

$$\frac{d^2\phi}{dx dy} + c e^\phi = 0,$$

then the solution will be

$$u = D^{-n} v_n$$

where

$$D = e^{-\phi} \frac{d}{dy},$$

where

$$v_n = \int e^{-\frac{\beta_n}{\Delta\beta_n} \phi} \chi y dy + \psi x,$$

χy and ψx arbitrary functions of y and x ,

and

$$\beta_n = \frac{\Delta\alpha_{n-1} \cdot \Delta\alpha_{n-2} \dots}{(\Delta\alpha_{n-1} - c)(\Delta\alpha_{n-2} - c) \dots},$$

where

$$\Delta\alpha_r = \alpha_{r+1} - \alpha_r,$$

and α_n is a function of n vanishing for $n=0$ and for $n=-1$.

I will proceed to demonstrate this curious theorem as briefly as possible.

According to the notation, we may write the given equation

$$D \frac{du}{dx} + \alpha_n u = 0.$$

Then if

$$v_n = D^n u$$

we have

$$v_{n+1} = D^{n+1} u = D v_n.$$

Suppose that

$$D \frac{dv_n}{dx} + \beta_n z D v_n = D^n \left\{ D \frac{du}{dx} + \alpha_n u \right\}. \quad (\alpha.)$$

where z is a function of x and y to be determined, also β_n a function of u .

Writing $n+1$ for n in equation $(\alpha.)$, we ought to have

$$D \frac{dv_{n+1}}{dx} + \beta_{n+1} z D v_{n+1} = D^{n+1} \left\{ D \frac{du}{dx} + \alpha_{n+1} u \right\} \quad (\beta.)$$

This circumstance will serve to determine z and β_n as follows: we have identically

$$\begin{aligned} D^{n+1}\left\{D \frac{du}{dx} + \alpha_{n+1}u\right\} &= D^{n+1}\left\{D \frac{du}{dx} + \alpha_n u\right\} + \Delta\alpha_n D^{n+1}u \\ &= D^2 \frac{dv_n}{dx} + \beta_n D\{z Dv_n\} + \Delta\alpha_n \cdot Dv_n \text{ by } (\alpha) \\ &= D^2 \frac{dv_n}{dx} + \beta_n D\{zv_{n+1}\} + \Delta\alpha_n \cdot v_{n+1}. \end{aligned}$$

But

$$\begin{aligned} \frac{dv_{n+1}}{dx} &= \frac{d}{dx} Dv_n = \frac{d}{dx} \left(e^{-\phi} \frac{dv_n}{dy} \right) \\ &= -\frac{d\phi}{dx} e^{-\phi} \frac{dv_n}{dy} + e^{-\phi} \frac{d}{dy} \cdot \frac{dv_n}{dx} \\ &= -\frac{d\phi}{dx} Dv_n + D \frac{dv_n}{dx}; \end{aligned}$$

$$\therefore D^2 \frac{dv_n}{dx} = D \left\{ \frac{dv_{n+1}}{dx} + \frac{d\phi}{dx} v_{n+1} \right\}.$$

Hence

$$D \frac{dv_{n+1}}{dx} + D \left(\frac{d\phi}{dx} v_{n+1} \right) + \beta_n D(zv_{n+1}) + \Delta\alpha_n v_{n+1}$$

ought to be identical with $D \frac{dv_{n+1}}{dx} + \beta_{n+1} z Dv_{n+1}$, and hence the conditions

$$\begin{aligned} \frac{d\phi}{dx} + \beta_n z &= \beta_{n+1} z \\ D \left\{ \frac{d\phi}{dx} + \beta_n z \right\} &= -\Delta\alpha_n. \end{aligned}$$

Eliminating z , we have

$$D \frac{d\phi}{dx} = -\frac{\Delta\alpha_n \Delta\beta_n}{\beta_{n+1}} = -c,$$

or

$$\frac{d^2\phi}{dx dy} + c e^{\phi} = 0,$$

and

$$c\beta_{n+1} = \Delta\alpha_n \{\beta_{n+1} - \beta_n\},$$

or

$$\beta_{n+1} = \frac{\Delta\alpha_n}{\Delta\alpha_n - c} \cdot \beta_n;$$

$$\therefore \beta_n = \frac{\Delta\alpha_{n-1} \cdot \Delta\alpha_{n-2} \dots}{(\Delta\alpha_{n-1} - c)(\Delta\alpha_{n-2} - c) \dots};$$

by these determinations we establish the formula (β .) as a consequence of (α .), and therefore if the formula (α .) be true for any value of n , it will be (subject to the above conditions) true for the next superior value.

Now, when $n=0$, $v_0 = D^0 u = u$, and provided α_0 and α_{-1} are each $=0$, α_0 and $\Delta\alpha_{-1}$ will be each 0, and $\therefore \alpha_0$ and β_0 each $=0$, and the equation (α .) reduces to $D \frac{dv_0}{dx}$

$= D \frac{du}{dx}$, and is therefore true for $u=0$. Under these restrictions it will therefore be true for any positive integral value of n . Now the symbol D represents $e^{-\phi} \frac{d}{dy}$, and therefore if $U=0$, $D^n U=0$, so that we have

$$D \frac{dv_n}{dx} + \beta_n x D v_n = 0,$$

or
$$e^{-\phi} \frac{d^2 v_n}{dx dy} + \frac{\beta_n}{\Delta \beta_n} \cdot \frac{d\phi}{dx} \cdot e^{-\phi} \frac{dv_n}{dy} = 0,$$

or
$$\frac{\frac{d}{dx} \cdot \frac{dv_n}{dy}}{\frac{dv_n}{dy}} = - \frac{\beta_n}{\Delta \beta_n} \cdot \frac{d\phi}{dx};$$

\therefore integrating with respect to x ,

$$\frac{dv_n}{dy} = e^{-\frac{\beta_n}{\Delta \beta_n} \phi} \chi y$$

$$v_n = \int e^{-\frac{\beta_n}{\Delta \beta_n} \phi} \chi y dy + \psi x,$$

and

$$u = D^{-n} v_n = \int e^{\phi} \int e^{\phi} \dots v_n dy dy \dots$$

the integral sign repeated n times. The theorem is therefore demonstrated.

It may be easily shown that the equation of LAPLACE's coefficients is included in the class here considered.

The equation of LAPLACE by a proper choice of independent variables assumes the form

$$\frac{d^2 u}{dx dy} + \frac{n \cdot n + 1}{4 \cos^2 \frac{y-x}{2}} \cdot u = 0.$$

Hence with reference to the preceding investigation,

$$D = \cos^2 \frac{y-x}{2} \cdot \frac{d}{dy} \text{ and } \alpha_n = \frac{n \cdot n + 1}{4}.$$

Hence
$$e^{-\phi} = \cos^2 \frac{y-x}{2};$$

$\therefore \frac{d\phi}{dx} = - \tan \frac{y-x}{2}$

$$\frac{d^2 \phi}{dx dy} + \frac{1}{2} e^{\phi} = 0.$$

Hence
$$c = \frac{1}{2}. \quad \text{Also } \Delta \alpha_n = \frac{n+1}{2}; \quad \Delta \alpha_n - c = \frac{n}{2};$$

$\therefore \beta_n = n \text{ and } \Delta \beta_n = 1.$

Inserting these values in the final formula, we have

$$v_n = \int \cos^{2n} \frac{y-x}{2} \chi y dy + \psi x,$$

and

$$u = \int \cos^{-2} \frac{y-x}{2} \int \cos^{-2} \frac{y-x}{2} \dots v_n dy dy \dots n \text{ times},$$

which agrees with Mr. HARGREAVE'S solution.